Gauss’s Law

Chapter 24

Gauss’s Law and Electric Flux

Gauss’s law is based on the concept of flux:

\[ \Phi = \int \mathbf{E} \cdot d\mathbf{A} \]

You can think of the flux through some surface as a measure of the number of field lines which pass through that surface. Flux depends on the strength of \( \mathbf{E} \), on the surface area, and on the relative orientation of the field and surface.

Electric flux

The flux also depends on orientation:

\[ \Phi = E \cdot A \cdot \cos \theta \]

The number of field lines through the tilted surface equals the number through its projection. Hence the flux through the tilted surface is simply given by the flux through its projection: \( E(A \cos \theta) \).

Here flux \( \Phi = E \cdot A \cos \theta = E \cdot A \cdot \cos \theta \).

But what if the electric field is not constant? What if it varies (possibly in both magnitude and direction) as a function of \( r \)?

This possibility is sketched here for the case of a closed surface.

Electric flux

But what if the electric field is not constant? What if it varies (possibly in both magnitude and direction) as a function of \( r \)?

This possibility is sketched here for the case of a closed surface.

We have to sum all the \( d\Phi \)'s over the entire surface: \( \Phi = \sum d\Phi \).

For accuracy the \( dA \)'s and thus the \( d\Phi \)'s must be very small.
Electric flux

But what if the electric field is not constant? What if it varies (possibly in both magnitude and direction) as a function of $\mathbf{E}$? This possibility is sketched here for the case of a closed surface.

$\Phi = \lim_{\Delta A \to 0} \sum_{\text{closed surface}} d\Phi = \lim_{\Delta A \to 0} \sum_{\text{closed surface}} \mathbf{E} \cdot d\mathbf{A} = \oint_{\text{closed surface}} \mathbf{E} \cdot d\mathbf{A}$

The loop means the integral is over a closed surface.

Gauss’s Law

Hence, Gauss’ Law states:

$$\Phi = \oint_{S} \mathbf{E} \cdot d\mathbf{A} = \frac{\sum q_{\text{inside}}}{\varepsilon_{0}}$$

This is always true. It’s sometimes useless, but often a very easy way to find the electric field (for highly symmetric cases).

Apply Gauss’ s law to a point charge

Consider a positive point charge $q$. Define a Gaussian surface (i.e. a closed surface) which is a sphere of radius $r$. By symmetry, the lines of $\mathbf{E}$ must be radially outwards, with magnitude depending only on $r$.

$$\oint_{S} \mathbf{E} \cdot d\mathbf{A} = \oint_{S} \mathbf{E} d\mathbf{A} = \frac{q}{\varepsilon_{0}}$$

using $\oint_{S} d\mathbf{A}$ = total area = $4\pi r^{2}$

gives $E = \frac{1}{4\pi \varepsilon_{0}} \frac{q}{r^{2}}$

→ Coulomb’s Law!

Is Gauss’s Law more fundamental than Coulomb’s Law?

- Maybe? Here we derived Coulomb’s law for a point charge from Gauss’s law.
- One can instead derive Gauss’s law for a general (even very nasty) charge distribution from Coulomb’s law. The two laws are equivalent.
- Gauss’s law gives us an easy way to solve very symmetric problems in electrostatics.
- Gauss’s law also gives us great insight into the electric fields in and on conductors and within voids inside metals.
- Gauss’s law has applications in electricity, magnetism, and even gravity. Mathematically, it applies fundamentally to vector fields and their potentials. Mathematically, Gauss’s law is very fundamental.

Symmetry

- Apply Gauss’s Law to a point charge and what do you get? Answer: Coulomb’s Law!
- We used the fact that a point charge in space is spherically symmetric.
- Gauss’s Law is always true, but is only useful for problems with usable symmetry.

Gauss’s Law

The total flux within a closed surface is proportional to the enclosed charge.

$$\Phi = \oint_{S} \mathbf{E} \cdot d\mathbf{A} = \frac{Q_{\text{enclosed}}}{\varepsilon_{0}}$$

Gauss’s Law is always true, but is only useful for problems with usable symmetry.
Can we figure out how the field varies with distance from the field lines and the symmetry?

Look at a point charge:
- How do its field lines look?

Field lines point out (or in) radially in all directions (3D)

Recall that the magnitude of $E$ is related to the density of field lines per unit area.

How does the number of field lines per unit area vary with distance?
- Inverse-square law: $E \propto \frac{1}{r^2}$

The lines spread in 2 directions.

Now how does flux density vary with distance?

The lines spread in 1 direction. In this case only the vertical direction.
**Symmetry and the Electric Field**

*Sheet of charge:*

Now how does the field change with distance?

- Field is a constant! (If sheet is infinite.)

**Applications of Gauss’s Law**

_Gauss’s Law does what we just did above, but does it rigorously._

We are now going to look at various charged objects and use Gauss’s law to find the field distribution.

**Problem: Sphere of Charge Q**

A charge Q is uniformly distributed through a sphere of radius R. What is the electric field as a function of r? Find E at r₁ and r₂.

Use symmetry!

This is spherically symmetric. That means that E(r) is radially outward, and that all points at a given radius (|r|=r) have the same magnitude of field.

First find \( E(r) \) at a point outside the charged sphere. Apply Gauss’s law, using as the Gaussian surface the sphere of radius r pictured.

What is the enclosed charge? Q
**Problem: Sphere of Charge Q**

First find $\mathbf{E}(r)$ at a point outside the charged sphere. Apply Gauss’s law, using as the Gaussian surface the sphere of radius $r$ pictured.

What is the enclosed charge? $Q$

What is the flux through this surface? $\Phi = \int \mathbf{E} \cdot d\mathbf{A} = \int \mathbf{E} dA$

$= \frac{1}{2} E \cdot 4\pi r^2 = 2\pi r^2 E$

Gauss: $\Phi = Q_{\text{enclosed}} / \varepsilon_0 = Q / \varepsilon_0$

$Q / \varepsilon_0 = \Phi = E(4\pi r^2)$

So $E(r) = \frac{1}{4\pi \varepsilon_0} \frac{Q}{r^2}$

**Problem: Sphere of Charge Q**

Look closer at these results. The electric field at $r$ comes from a sum over the contributions of all the little bits.

$E(r)$ is proportional to $r$ for $r < R$

$E(r)$ is proportional to $1/r^2$ for $r > R$

and $E(r)$ is continuous at $R$

**Problem: Sphere of Charge Q**

Next find $\mathbf{E}(r)$ at a point inside the sphere. Apply Gauss’s law, using a little sphere of radius $r$ as a Gaussian.

What is the enclosed charge? $Q_{\text{enc}}$

That takes a little effort. The little sphere has some fraction of the total charge. What fraction?

That’s given by volume ratio: $Q_{\text{enc}} = \frac{r^3}{R^3} Q$

Again the flux is: $\Phi = E A = E(4\pi r^2)$

setting $\Phi = Q_{\text{enc}} / \varepsilon_0$ gives $E = \left(\frac{r^3}{R^3}\right) Q$

for $r < R$, $E(r) = \frac{Q}{4\pi \varepsilon_0 R^2} \frac{1}{r}$

**Problem: Sphere of Charge Q**

Consider an infinite plane with a constant charge density $\sigma$ (which is some number of Coulombs per square meter). What is $\mathbf{E}$ at a point a distance $z$ above the plane?

**Problem: Infinite charged plane**

Because of the spherical symmetry, the contributions from the bits outside the radius of the exactly cancel one another!

The field at $r$ is exactly what you would have if all the charge within the radius $r$ were concentrated to a point at the origin.

**Problem: Sphere of Charge Q**

Now look at an observation point inside the sphere. $r < R$

Because of the spherical symmetry, the contributions from the bits outside the radius of exactly cancel one another!

The field at $r$ is exactly what you would have if all the charge within the radius $r$ were concentrated to a point at the origin.

Doing this as a volume integral would be HARD. Gauss’s law is EASY.
Problem: Infinite charged plane

Consider an infinite plane with a constant charge density \( \sigma \) (which is some number of Coulombs per square meter). What is \( E \) at a point a distance \( z \) above the plane?

The electric field must point straight away from the plane (if \( \sigma > 0 \)). Maybe the magnitude \( E \) depends on \( z \), but the direction is fixed. And \( E \) is independent of \( x \) and \( y \).

Use symmetry!

Let the area of the top and bottom be \( A \).

Total charge enclosed by box = \( A\sigma \)

Gauss’s law then says that \( A\sigma/\varepsilon_0 = EA \) so that \( E = \sigma/2\varepsilon_0 \), outward.

This is constant everywhere in each half-space!

Notice that the area \( A \) canceled: this is typical!
A conductor is a material in which charges can move relatively freely.
Usually these are metals.
In a static condition, the charges placed on a conductor will have moved as far from each other as possible - they repel each other.
In a static situation, the electric field is zero everywhere inside a conductor.

Why is $E=0$ inside a conductor?
Because conductors are full of free electrons, roughly one per cubic Angstrom. These are free to move. If $E$ is nonzero in some region, then the electrons there feel a force $-eE$ and start to move.

In an electrostatics problem, the electrons adjust their positions until the force on every electron is zero (or else it would move!). That means when equilibrium is reached, $E=0$ everywhere inside a conductor.

Because $E=0$ inside, the inside is neutral.
Suppose there is an extra charge inside. Gauss’s law for the little spherical surface says there would be a nonzero $E$ nearby. But there can’t be, within a metal!
Consequently the interior of a metal is neutral. Any excess charge ends up on the surface.

**Problem: Charged coaxial cable**

This picture is a cross section of an infinitely long line of charge surrounded by an infinitely long cylindrical conductor. Find $E$.

This represents the line of charge. Say it has a linear charge density of $\lambda$ (some number of C/m$^2$).

This is the cylindrical conductor. It has inner radius $a$ and outer radius $b$.

Use symmetry! Clearly $E$ points straight out, and its amplitude depends only on $r$. 
Problem: Charged coaxial cable

First find $E$ at positions in the space inside the cylinder ($r<a$).

Choose as a Gaussian surface a cylinder of radius $r$ and length $L$.

What is the charge enclosed? $\lambda L$

What is the flux through the end caps? zero (cos90°)

What is the flux through the curved face? $E \times (\text{area}) = E(2\pi rL)$

Total flux $= E(2\pi rL)$

Gauss’s law: $E(2\pi rL) = \lambda L/\varepsilon_0$ so $E(r) = \lambda / 2\pi r \varepsilon_0$

Problem: Charged coaxial cable

Now find $E$ at positions within the cylinder ($a<r<b$).

There’s no work to do: within a conductor $E=0$.

Still, we can learn something from Gauss’s law.

Make the same kind of cylindrical Gaussian surface. Now the curved side is entirely within the conductor, where $E=0$; hence the flux is zero.

Thus the total charge enclosed by this surface must be zero.

Example Problem: Gauss’ Law for Gravity

“Gauss’ law for gravitation” is $\Phi_g = \oint g \cdot dA = -4\pi G m_{\text{enclosed}}$

In which $\Phi_g$ is the net flux of the gravitational field $g$ through a gaussian surface that encloses a mass ($m_{\text{enclosed}}$). The field $g$ is defined to be the acceleration of a test particle on which $m_{\text{enclosed}}$ exerts a gravitational force.

Calculate Newton’s Law of Gravitation from this.

Example Problem: Gauss’ Law for Gravity

$F = m_a = G \frac{m \cdot m}{r^2}$

$\Phi_g = \oint g \cdot dA = -4\pi G m_{\text{enclosed}}$

$\Phi_g = - \oint g \cdot dA$

$\Phi_g = - \oint g \cdot dA = -g \cdot \oint dA$

$= -g A = -g \left(4\pi r^2\right)$

$\mathbf{a} = \frac{-g A}{m} \rightarrow \text{accel.} = g = \frac{\Theta m r^2}{2} = \mathbf{a}$