DC electrical circuits

Chapter 28

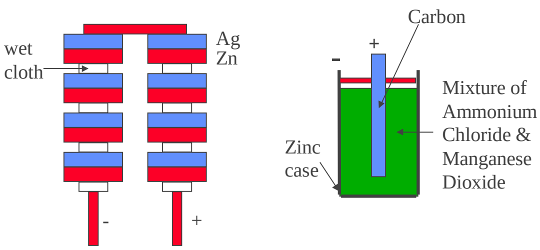
Electromotive Force
Potential Differences
Resistors in parallel and series
Circuits with Capacitors

Batteries and Generators

- Current is produced by applying a potential difference across a conductor (I=V/R).
- This potential difference is set up by some source such as a battery or generator.
- Conventionally an “applied voltage” is given the symbol E (units: volts).
- For historical reasons, this applied voltage is often called the “electromotive force” (emf), even though it’s not a force.

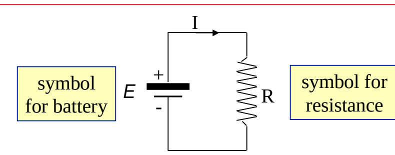
The Voltaic Pile

Volta’s original battery



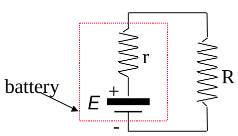
electrical converter...
.....converts chemical energy to electrical energy

Electrical description of a battery



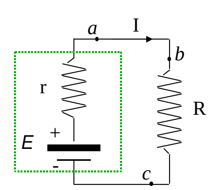
- A battery does work on positive charges in moving them to higher potential.
- The EMF E most precisely is the work per unit charge exerted to move the charges “uphill” ...
- ... but you can just think of it as an “applied voltage.”

Electrical description of a battery



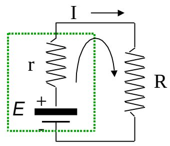
- One last point: Batteries actually have an internal resistance r. Often we neglect this, but sometimes it is important.

Current around a circuit



If the current at point a is I= 1 Ampere,
- what is the current at point b?
- ....at point c?

The Loop Method



Start at any point in the circuit.

Go around the circuit in a loop.

Add up the potential differences across each element. (Keep the signs straight!)

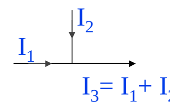
$$E - Ir - IR = 0$$ **(using V=IR)**

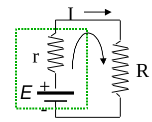
So $I = E/(R + r)$

Kirchhoff's Laws

Kirchhoff devised two laws that are universally applicable in circuit analysis:

1. At any circuit junction, currents entering must equal currents leaving (JUNCTION or NODE RULE).
2. Sum of all ΔV's across all circuit elements in a loop must be zero.(LOOP RULE).

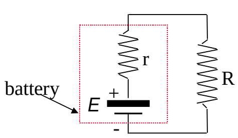




In general Kirchhoff's Laws are used in more complex circuits.

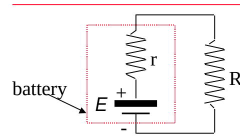
$$E - Ir - IR = 0$$

Example:



For a real battery with internal resistance r, what load R will receive a maximum power?

Example:



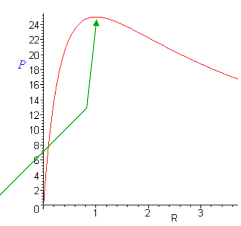
For a real battery with internal resistance r, what load R will receive a maximum power?

$P := iE$

$i := \frac{E}{r+R}$

$P := \frac{E^2 R}{(r+R)^2}$

Let $E := 10$ $r := 1$ *ie plotting*

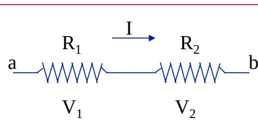


$\frac{dP}{dR} := -2\frac{E^2 R}{(r+R)^3} + \frac{E^2}{(r+R)^2}$

$\frac{dP}{dR} := 0$ *at* $R = r$

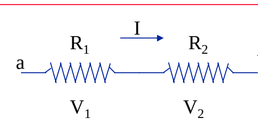
Thus $R_{maxPwr} := r$

Resistors in Series



The pair of resistors can be replaced by a single equivalent resistor; one which, given I, has the same total voltage drop as the original pair.

Resistors in Series



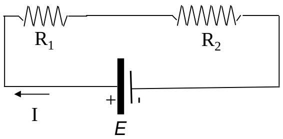
The pair of resistors can be replaced by a single equivalent resistor; one which, given I, has the same total voltage drop as the original pair.

$$V = V_1 + V_2 = I R_1 + I R_2 = I R_{eq}$$

We want to write this as $V = I R_{eq}$

hence $R_{eq} = R_1 + R_2$

Resistors in Series



Given $R_{eq} = R_1 + R_2$, the current is $I = E/(R_1 + R_2)$ (which agrees with the loop calculation)

One can then work backwards to get the voltage across each resistor:

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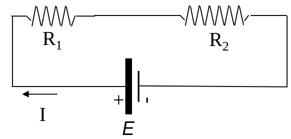
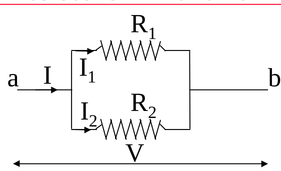


Given $R_{eq} = R_1 + R_2$, the current is $I = E/(R_1 + R_2)$ (which agrees with the loop calculation)

One can then work backwards to get the voltage across each resistor:

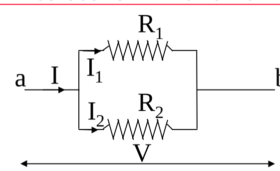
$$V_1 = IR_1 = E\frac{R_1}{R_1 + R_2} \qquad V_2 = IR_2 = E\frac{R_2}{R_1 + R_2}$$

Resistors in Parallel



Again find the equivalent single resistor which has the same V if I is given.

Resistors in Parallel



Again find the equivalent single resistor which has the same V if I is given. Here the total I splits:

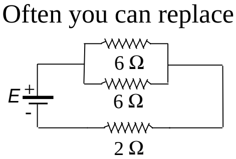
$I = I_1 + I_2 = V/R_1 + V/R_2 = V(1/R_1 + 1/R_2) = V/R_{eq}$

We want to write this as: $I = V/R_{eq}$

Hence $1/R_{eq} = 1/R_1 + 1/R_2$

Analyzing Resistor Networks

Often you can replace sets of resistors step by step.



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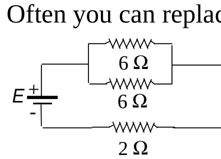
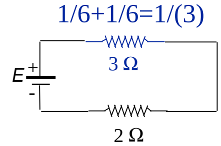
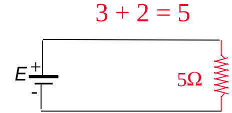


$1/6+1/6=1/(3)$



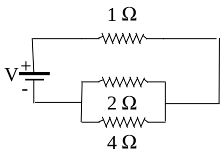
step 1

$3 + 2 = 5$



step 2

Find the current in the 2 Ω resistor if V = 20 volts



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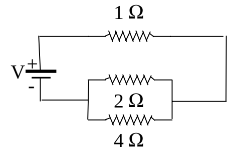
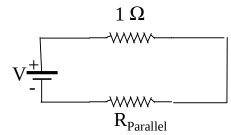


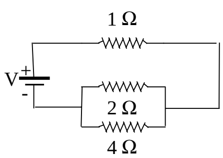
As before, find R_{eq} first.



$$\frac{1}{R_{parallel}} = \frac{1}{R_1} + \frac{1}{R_2} = \frac{1}{2\Omega} + \frac{1}{4\Omega} = \frac{3}{4\Omega}$$

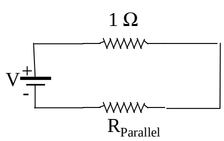
$$R_{parallel} = \frac{4}{3}\Omega$$

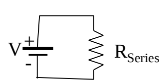
Find the current in the 2 Ω resistor if V = 20 volts



As before, find R_{eq} first.

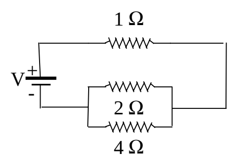
$$R_{parallel} = \frac{4}{3}\Omega = 1.33\Omega$$



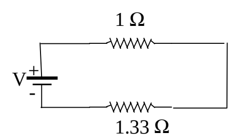


$R_{series} = 1\ \Omega + R_{parallel} = 2.33\ \Omega$

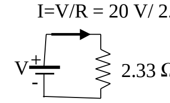
Find the current in the 2 Ω resistor if V = 20 volts



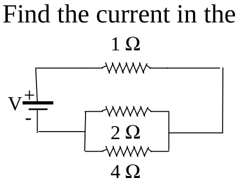
Now use V=IR and work back.



I=V/R = 20 V/ 2.33 Ω = 8.6 A



Find the current in the 2 Ω resistor if V = 20 volts



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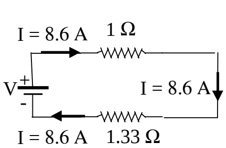

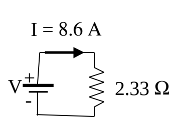

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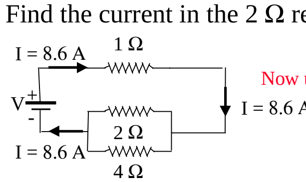

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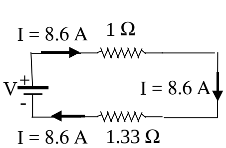

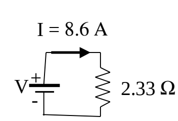

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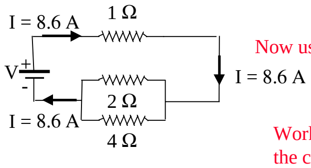

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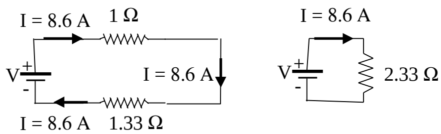
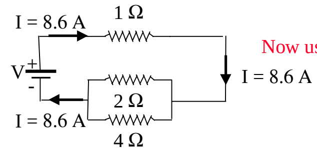
Working back in series the current is the same!!!



Find the current in the 2 Ω resistor if V = 20 volts



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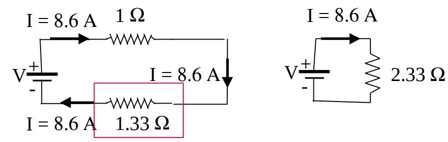
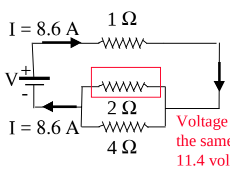


Voltage drop across the 1.33 Ω resistor is V = IR = (8.6 A)(1.33 Ω) = 11.4 volts

Find the current in the 2 Ω resistor if V = 20 volts



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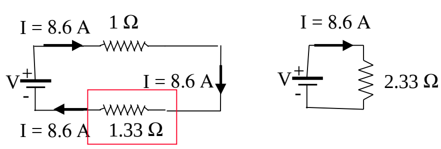
Voltage drop across the 2 Ω (and 4 Ω) resistor is the same as the 1.33 Ω resistor, thus 11.4 volts = $V_{2\Omega} = I_{2\Omega}R_{2\Omega}$



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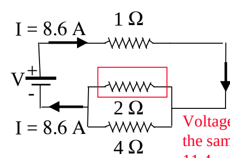

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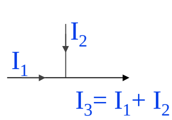
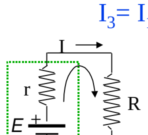
$$I_{2\Omega} = \frac{V_{2\Omega}}{R_{2\Omega}} = \frac{11.4V}{2\Omega} = 5.7\ amps$$

Kirchhoff's Laws

Kirchhoff devised two laws that are universally applicable in circuit analysis:

1. At any circuit junction, currents entering must equal currents leaving (JUNCTION or NODE RULE).
2. Sum of all ΔV's across all circuit elements in a loop must be zero.(LOOP RULE).





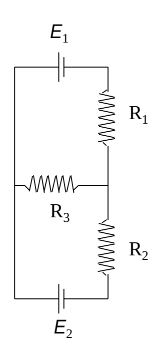
$E - Ir - IR = 0$

In general Kirchhoff's Laws are used in more complex circuits.

Example: Find the current in R_2

$R_1 = 20\Omega$
$R_2 = 5\Omega$
$R_3 = 30\Omega$

$E_1 = 25$ v
$E_2 = 5$ v



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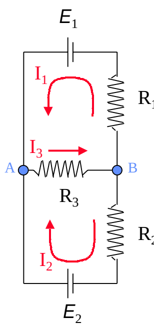
Define/guess current directions and nodes A and B.



Node Rule at A

$$I_1 + I_2 = I_3$$

Node Rule at B

$$I_3 = I_1 + I_2$$

Redundant Eq'ns! Thus only use one.

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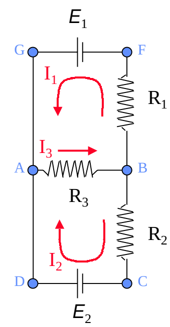

Node Rule at A

$$I_1 + I_2 = I_3$$

Loop Rule around ABCD

$$-I_3R_3 - I_2R_2 + E_2 = 0$$

Loop Rule around ABFG

$$-I_3R_3 - I_1R_1 + E_1 = 0$$

Example: Find the current in R_2

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$R_2 = 5\Omega$
$R_3 = 30\Omega$

$E_1 = 25$ v
$E_2 = 5$ v

$$I_1 + I_2 = I_3$$

$$-I_3R_3 - I_2R_2 + E_2 = 0$$

$$-I_3R_3 - I_1R_1 + E_1 = 0$$

Three eq'ns three unknowns (I_1, I_2, and I_3). We are looking for I_2.

Example: Find the current in R_2

$a := I_1 + I_2 = I_3$

$b := -I_3 R_3 - I_2 R_2 + E_2 = 0$

$c := -I_3 R_3 - I_1 R_1 + E_1 = 0$

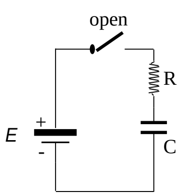
$$d := \{I_2 = \frac{-R_3 E_1 + E_2 R_3 + R_1 E_2}{R_3 R_2 + R_1 R_3 + R_1 R_2}, I_1 = \frac{R_3 E_1 - E_2 R_3 + E_1 R_2}{R_3 R_2 + R_1 R_3 + R_1 R_2}, I_3 = \frac{R_1 E_2 + E_1 R_2}{R_3 R_2 + R_1 R_3 + R_1 R_2}\}$$

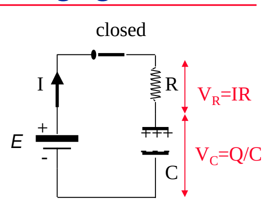
$$I\_2 := I_2 = \frac{-R_3 E_1 + E_2 R_3 + R_1 E_2}{R_3 R_2 + R_1 R_3 + R_1 R_2}$$

$$I_2 = \frac{-10}{17} = -.5882352941$$

$I_2 = -0.59$ amps
Negative implies the direction is opposite from what was assumed!

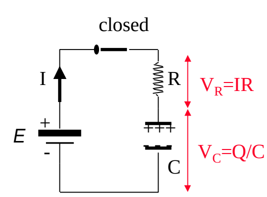
RC circuits: charging





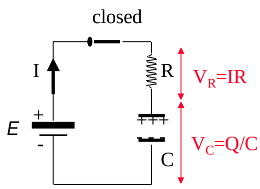
When the switch closes, at first a high current flows: V_R is big and V_C is small.
As q is stored in C, V_C increases. This fights against the battery and so I decreases.

RC circuits: charging



Apply the loop law:
$E - IR - Q/C = 0$

RC circuits: charging



Apply the loop law:
E-IR-Q/C=0

$$E - R\frac{dQ}{dt} - \frac{1}{C}Q = 0$$

Now it's a math problem!

$$E - R\frac{dQ}{dt} - \frac{1}{C}Q = 0,\quad E - \frac{Q}{C} = R\frac{dQ}{dt}$$

$$dt = R\frac{dQ}{E - \frac{Q}{C}} \qquad (1)$$

$$x = E - \frac{Q}{C},\quad dx = -\frac{dQ}{C},\quad -C dx = dQ$$

$$(1)\ \textit{becomes}\quad dt = -RC\frac{dx}{x},\ \rightarrow \int dt = -RC\int\frac{dx}{x}$$

$$t + c_1 = -RC\ \ln(x) = -RC\ \ln(E - \frac{Q}{C}),\quad (c_1 = \text{const. of int.})$$

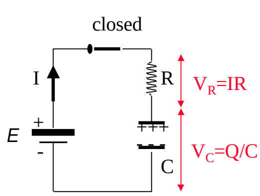
$$E - \frac{Q}{C} = c_2 e^{-\frac{t}{RC}},\quad (c_2 = e^{c_1}),\quad Q(t) = CE - c_3 e^{-\frac{t}{RC}},\quad (c_3 = C\ c_2)$$

$$Q(t) = CE(1 - c_4 e^{-\frac{t}{RC}}),\quad (c_4 = \frac{c_3}{CE})$$

Boundary Condition $Q(t=0) = 0$, *so* $c_4 = 1$ *and* $c_1 = \ln(E)$

Thus, $Q(t) = CE(1 - e^{-\frac{t}{RC}})$

RC circuits: charging



Apply the loop law:
E-IR-Q/C=0

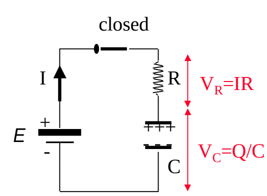
$$E - R\frac{dQ}{dt} - \frac{1}{C}Q = 0$$

Solving we get: $Q = CE(1 - e^{-\frac{t}{RC}})$

$$I = \frac{dQ}{dt} = \frac{E}{R}e^{-\frac{t}{RC}}$$

Where this assumes that at t=0 sec the switch is closed and there was no charge on the CAP before t=0 sec.
Q=charge on CAP & I=current in circuit.

RC circuits: charging



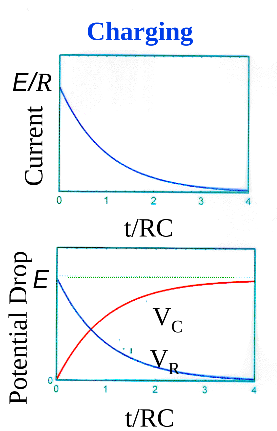
$$Q = CE(1 - e^{-\frac{t}{RC}})$$

$$I = \frac{\Delta Q}{\Delta t} = \frac{E}{R}e^{-\frac{t}{RC}}$$

From this we get:
With the same assumptions at t=0 sec.

$$V_R = IR = R\frac{E}{R}e^{-\frac{t}{RC}} = Ee^{-\frac{t}{RC}}$$

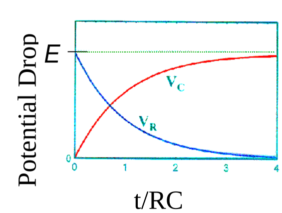
$$V_C = E - V_R = E\left(1 - e^{-\frac{t}{RC}}\right)$$



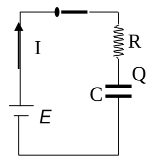
Note: the plots have t/RC along the x-axis.

RC has units of time!

RC = τ, where τ is the time constant of the circuit.



Example: A capacitor C charges through a resistor R. When does its charge rise to half its final value ?



Example: A capacitor C charges through a resistor R. When does its charge rise to half its final value ?

Charge on a capacitor varies as

$$Q = CE(1 - e^{-\frac{t}{RC}})$$

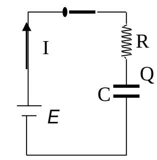
Find the time for which Q=CE/2

$$\frac{CE}{2} = CE\,(1 - e^{-\frac{t}{RC}})$$

$$\frac{1}{2} = (1 - e^{-\frac{t}{RC}})$$

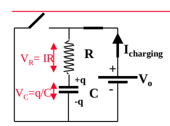
$$\frac{1}{2} = e^{-\frac{t}{RC}} = \frac{1}{e^{\frac{t}{RC}}}, \quad 2 = e^{\frac{t}{RC}}$$

$$\ln(2) = \frac{t}{RC}, \quad t = \ln(2)\,RC \cong 0.69\,RC$$



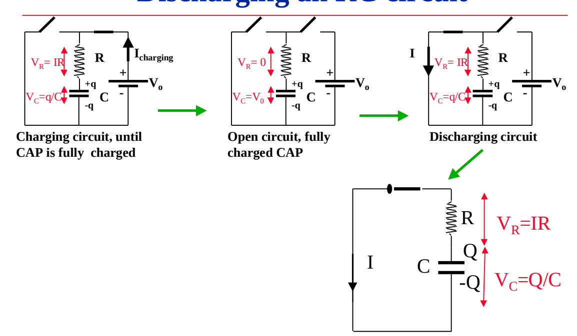
RC is the "time constant"

Discharging an RC circuit



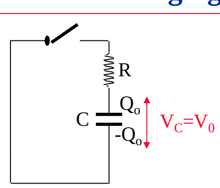
Charging circuit, until CAP is fully charged

Discharging an RC circuit

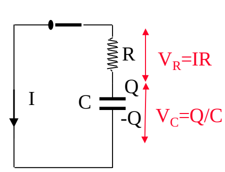


After closing switch
Assume switch is closed at t=0 s.

Discharging an RC circuit



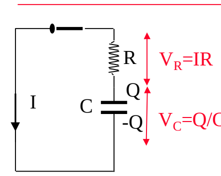
Open circuit



After closing switch

Current will flow through resistor for a while. Power $P = IV = I^2R$ will be dissipated in the resistor (as heat) while the current flows. Eventually the capacitor will lose its charge, and the current will go to zero.

Discharging an RC circuit



Loop equation:
Q/C - IR = 0

$$\frac{Q}{C} - \frac{dQ}{dt}R = 0$$

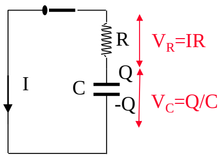
$$Q = Q_o e^{-\frac{t}{RC}}$$

Q_o = the amount of charge on the CAP at t=0 sec.

And $$I = \frac{Q_o}{RC} e^{-t/RC}$$

I = the current in the circuit at any time after t=0 sec.

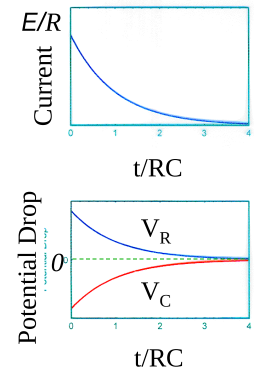
Discharging an RC circuit



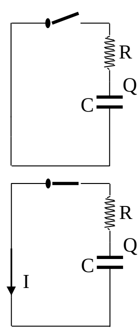
$$V_R = IR = R\frac{Q_o}{RC}e^{-t/RC} = \frac{Q_o}{C}e^{-t/RC}$$

$$V_C = \frac{Q}{C} = -\frac{Q_o e^{-\frac{t}{RC}}}{C} = -\frac{Q_o}{C}e^{-\frac{t}{RC}}$$

Discharging



Example: A capacitor C discharges through a resistor R.
(a) When does its charge fall to half its initial value ?



Example: A capacitor C discharges through a resistor R.
(a) When does its charge fall to half its initial value ?

Charge on a capacitor varies as

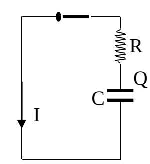
$$Q = Q_0 \exp(-t/RC)$$

Find the time for which $Q=Q_0/2$

$$\frac{1}{2}Q_0 = Q_0 \exp(-t/RC)$$

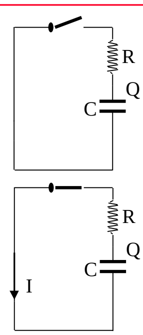
$$\ln\frac{1}{2} = -\ln 2 = -\frac{t}{RC}$$

$$t = (\ln 2)RC = 0.69RC$$



RC is the "time constant"

Example: A capacitor C discharges through a resistor R.
(b) When does the energy drop to half its initial value?



Example: A capacitor C discharges through a resistor R.
(b) When does the energy drop to half its initial value?

The energy stored in a capacitor is

$$U(t) = \frac{Q^2}{2C} = \frac{Q_0^2}{2C}\exp(-2t/RC) = U_0 \exp(-2t/RC)$$

We seek the time for U to drop to $U_0/2$:

$$\frac{1}{2}U_0 = U_0 \exp(-2t/RC)$$

$$\therefore t = RC\frac{\ln 2}{2} = 0.35RC$$