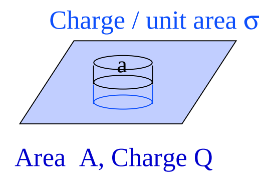
Electrostatic Energy and Capacitance & Dielectrics

Chapter 26

Energy of two isolated conductors

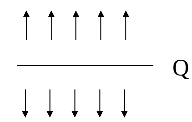
Consider a large charged conducting plane: $\sigma = Q/A$



Find E using Gauss' law.
E points straight away both above and below the surface.

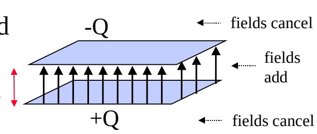
Hence $\Phi = 2Ea$, while $Q_{enc} = \sigma\, a$.

Gauss's law says $\Phi = Q_{enc}/\varepsilon_0$.
Thus $E = Q / 2A\varepsilon_0$.



Energy of two isolated conductors

Now we bring a second plate with charge -Q to within distance d



E* = *Q / (A* ε_0*) between the plates

Therefore potential difference between the plates is $V = E\,d = Q\,d/(\varepsilon_o A)$

Relationship between V and Q: $Q = CV, \quad C = \frac{\varepsilon_o A}{d}$

C is called "Capacitance".
(Units: 1Coulomb /Volt = 1 Farad)

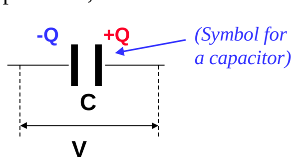
What does a Capacitor do?

- Stores electrical charge.
- Stores electrical energy.

What does a Capacitor do?

- Stores electrical charge.
- Stores electrical energy.

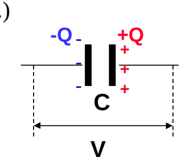
The charge is easy to see. If a certain potential, V, is applied to a capacitor C, it must store a charge **Q=CV**:



What does a Capacitor do?

- Stores electrical charge.
- Stores electrical energy.

As to how the energy is stored, there is a <u>separation</u> of charge. (The <u>net</u> charge on the capacitor is **zero.**)



Energy stored in a capacitor

- Suppose we have a capacitor with charge q (+ and -).
- Then we increase the charge by dq (+ and -).
- We must do work dW = Vdq to increase charge:

Energy stored in a capacitor

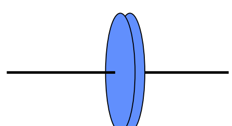
- Suppose we have a capacitor with charge q (+ and -).
- Then we increase the charge by dq (+ and -).
- We must do work dW = Vdq to increase charge:

$$i.e. \quad dW = \frac{q}{C}dq$$

- Integrating q from 0 to Q, we can find total stored (potential) electric energy: $U = W = \int_o^Q dW = \int_0^Q \frac{q}{C}dq$
- Therefore the stored energy is: $U = W = \frac{Q^2}{2C} = \frac{1}{2}CV^2$

A P.P. CAP is constructed with circular plate of radius 10.0 cm and separated by 1.0 mm with a potential difference between the plates of 100 V. Find the:

a) capacitance,
b) electric field, and
c) energy stored in the CAP.



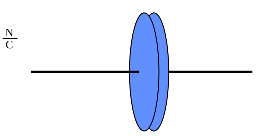
$$C = \frac{\varepsilon_o A}{d} = \frac{\varepsilon_o \pi r^2}{d}$$

$$= \frac{8.85\times10^{-12}\,\frac{C^2}{Nm^2}\,\pi(0.1m)^2}{0.001\,m} = 2.78\times10^{-10}\,F$$

A P.P. CAP is constructed with circular plate of radius 10.0 cm and separated by 1.0 mm with a potential difference between the plates of 100 V. Find the:

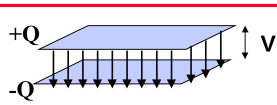
a) capacitance,
b) electric field, and
c) energy stored in the CAP.

$$V = Ed, E = \frac{V}{d} = \frac{100\ V}{0.001\ m} = 10^5\ \frac{N}{C}$$



$$U = \frac{1}{2}CV^2 = \left(\frac{1}{2}\right)2.78\times10^{-10}F(100\,V)^2 = 5.56\times10^{-6}J$$

Energy density.

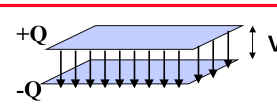


- Now compute the energy density, u_E, inside the capacitor.
- For a parallel plate capacitor of volume A•d,

$u_E = U/(Ad) = (1/2\ CV^2)/Ad$

But for a parallel plate capacitor, **$C = \varepsilon_0 A/d$**

Energy density.



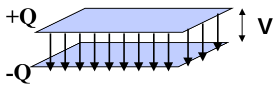
- Now compute the energy density, u_E, inside the capacitor.
- For a parallel plate capacitor of volume A.d,

$u = U/(Ad) = (1/2\ CV^2)/Ad$

But for a parallel plate capacitor, **$C = \varepsilon_0 A/d$**

→ **$u_E = (\varepsilon_0/2)(V/d)^2 = (\varepsilon_0/\ 2)E^2$**

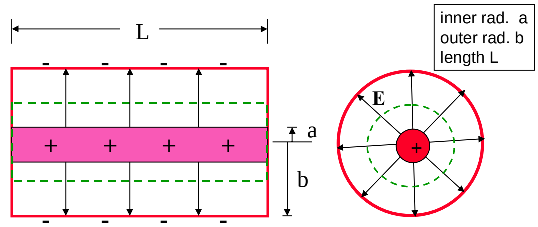
Energy density.



$u_E = (\varepsilon_0/2)(V/d)^2 = (\varepsilon_0/2)E^2$

- This leads to another understanding of electric field
- The energy is stored in the FIELD, rather than in the plates!
- If an electric field exists, then you can associate an electric potential energy of $(\varepsilon_0/2)E^2$

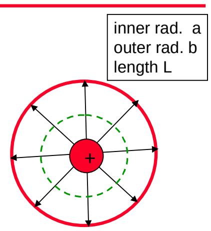
Cylindrical capacitor



By symmetry the electric field must point outward

Cylindrical capacitor

Construct Gaussian surface around inner cylinder. - - - - -

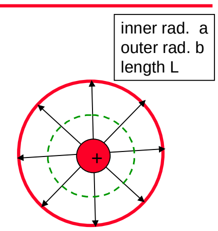


Cylindrical capacitor

Construct Gaussian surface around inner cylinder. - - - - -

Then $q/\varepsilon_0 = EA = E(2\pi rL)$

Therefore $E = q/(\varepsilon_0 2\pi rL)$



Cylindrical capacitor

Construct Gaussian surface around inner cylinder. - - - - -

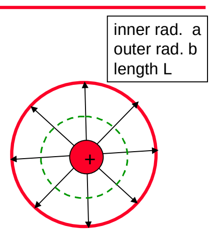
Then $q/\varepsilon_0 = EA = E(2\pi rL)$

Therefore $E = q/(\varepsilon_0 2\pi rL)$

$$V = -\int_i^f \vec{E} \bullet d\vec{r} = \int_+^- E dr$$

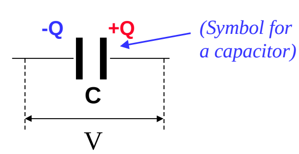
$$= \frac{q}{2\pi\varepsilon_0 L}\int_a^b \frac{dr}{r} = \frac{q}{2\pi\varepsilon_0 L}\ln\left(\frac{b}{a}\right)$$

$$C = 2\pi\varepsilon_0 \frac{L}{\ln(b/a)}$$ in Farads



Note: C is made larger by making (b - a) as small as possible.

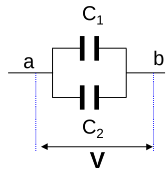
Capacitors in circuits



A piece of metal in equilibrium has a constant value of potential everywhere. So the potential of a plate and attached wire is the same. The potential difference between the ends of the wires is V, the same as the potential difference between the plates.

Capacitors in parallel

- It turns out that you could replace these by a single capacitor of capacitance C_{eq}.
- Suppose there is a potential difference V between a and b.
- Then $q_1 = C_1V$ & $q_2 = C_2V$

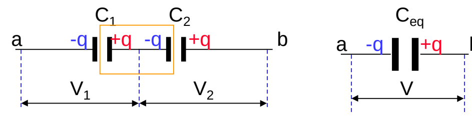


- Hence the total charge to flow in through a wire is $q = q_1 + q_2$
- Try to seek an equivalent capacitance for which $q = C_{eq}$ V:
  that is, $C_{eq}V = C_1 V + C_2 V$. Cancelling V gives

$$C_{eq} = C_1 + C_2$$

- This is the equation for capacitors in parallel.
- Increasing the number of capacitors increases the capacitance.

Capacitors in series



- Here the total potential difference between a and b is $V = V_1 + V_2$
- The charge on every plate of the capacitors must be the same in magnitude.
- We seek $V = q/ C_{eq} = q/ C_1 + q/ C_2$. Cancelling q gives:

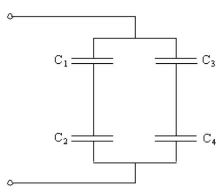
$$1 / C_{eq} = 1 / C_1 + 1 / C_2$$

- This is the equation for capacitors in series.
- Increasing the number of capacitors decreases the capacitance.

Example

Find the equivalent capacitance.

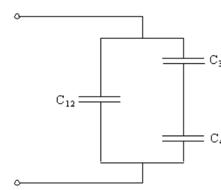
$C_1 = 1\ \mu F$
$C_2 = 2\ \mu F$
$C_3 = 3\ \mu F$
$C_4 = 1\ \mu F$



Example

Find the equivalent capacitance.

$C_1 = 1\ \mu F$
$C_2 = 2\ \mu F$
$C_3 = 3\ \mu F$
$C_4 = 1\ \mu F$

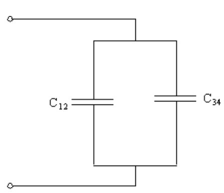


$$\frac{1}{C_{12}} = \frac{1}{C_1} + \frac{1}{C_2}, C_{12} = 0.67\mu F$$

Example

Find the equivalent capacitance.

$C_1 = 1\ \mu F$
$C_2 = 2\ \mu F$
$C_3 = 3\ \mu F$
$C_4 = 1\ \mu F$



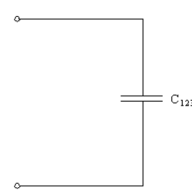
$$\frac{1}{C_{12}} = \frac{1}{C_1} + \frac{1}{C_2}, C_{12} = 0.67\mu F$$

$$\frac{1}{C_{34}} = \frac{1}{C_3} + \frac{1}{C_4}, C_{34} = 0.75\mu F$$

Example

Find the equivalent capacitance.

$C_1 = 1\ \mu F$
$C_2 = 2\ \mu F$
$C_3 = 3\ \mu F$
$C_4 = 1\ \mu F$



$$\frac{1}{C_{12}} = \frac{1}{C_1} + \frac{1}{C_2}, C_{12} = 0.67\mu F$$

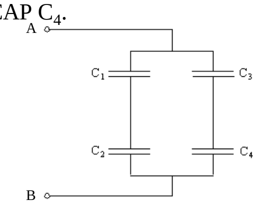
$$\frac{1}{C_{34}} = \frac{1}{C_3} + \frac{1}{C_4}, C_{34} = 0.75\mu F$$

$$C_{1234} = C_{12} + C_{34} = 1.42\mu F$$

Example

If 10 volts of potential exist between points A and B, what is the charge on CAP C_4.

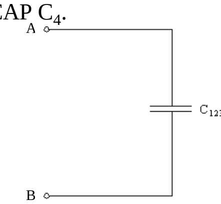
$C_1 = 1\ \mu F$
$C_2 = 2\ \mu F$
$C_3 = 3\ \mu F$
$C_4 = 1\ \mu F$



Example

If 10 volts of potential exist between points A and B, what is the charge on CAP C_4.

$C_1 = 1\ \mu F$
$C_2 = 2\ \mu F$
$C_3 = 3\ \mu F$
$C_4 = 1\ \mu F$



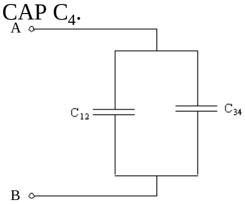
$Q = CV$

$Q_{1234} = C_{1234} V_{AB}$
$\approx 1.42\ \mu F\ 10\ V$
$\approx 0.142\ mC$

Example

If 10 volts of potential exist between points A and B, what is the charge Q_4 on CAP C_4.

$C_1 = 1\ \mu F$
$C_2 = 2\ \mu F$
$C_3 = 3\ \mu F$
$C_4 = 1\ \mu F$



$Q_{1234} = Q_{12} + Q_{34}$

But more significant is

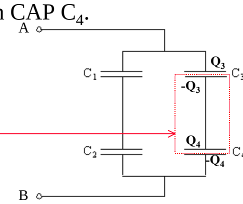
$Q_{34} = C_{34} V_{AB}$
$\approx 0.75\ \mu F\ 10\ V$
$\approx 75\ \mu C$

Thus, the calculation of Q_{1234} in the previous step was not required. And we're done because $Q_4 = Q_{34}$, but we'll do it to see why!

Example

If 10 volts of potential exist between points A and B, what is the charge Q_4 on CAP C_4.

$C_1 = 1\ \mu F$
$C_2 = 2\ \mu F$
$C_3 = 3\ \mu F$
$C_4 = 1\ \mu F$



Total charge inside the red box is zero, so Q_3 must equal Q_4. Furthermore, Q_3 is the same plate that Q_{34} was so $Q_3 = Q_{34}$.

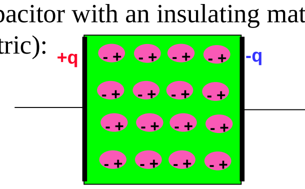
Thus, we can conclude that $Q_4 = Q_{34} \approx 75\ \mu C$

Dielectrics in Capacitors

- Suppose we fill the space between the plates of a capacitor with an insulating material (a "dielectric):

Dielectrics in Capacitors

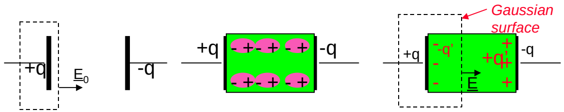
- Suppose we fill the space between the plates of a capacitor with an insulating material (a "dielectric):



The material will be "polarized" - electrons are pulled away from atom cores

- Consequently the E field within the capacitor is reduced

Dielectrics in Capacitors



- Calculate this by Gauss' Law, noting there is an induced charge q' on surface of dielectric.

$$\varepsilon_o \oint \vec{E} \bullet d\vec{A} = \varepsilon_o EA = (q - q')$$

- The field reduced by factor κ_e: $E = E_0/\kappa$.
  Here κ is the *dielectric constant.*

Effect on Capacitance

- A dielectric reduces the electric field by a factor κ
- Hence $V = E\, d$ is reduced by κ
- Therefore $C = Q/V$ is increased by κ

$$C = \frac{\varepsilon_o \kappa A}{d}$$

- parallel plate capacitor with dielectric.

- Adding a dielectric *increases* the capacitance.

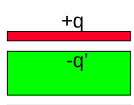
Dielectrics & Gauss's Law

With a dielectric present, Gauss's Law can be rewritten from

$$\varepsilon_0 \oint \underline{E} \bullet \underline{dA} = q - q'$$

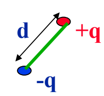
to

$$\varepsilon_0 \oint \kappa \underline{E} \bullet \underline{dA} = q$$



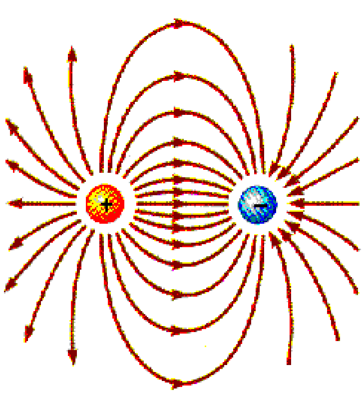
Instead of having to think about the confusing induced charge q', we can simply use the free charge q. But E is replaced by κE.

The electric dipole

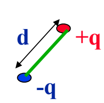


An electric dipole consists of two equal and opposite charges (q and -q) separated a distance d. In a dipole the separation distance is held fixed.

Field Due to an Electric Dipole



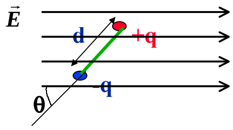
The electric dipole



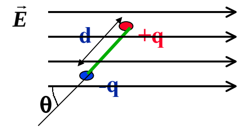
An electric dipole consists of two equal and opposite charges (q and -q) separated a distance d. In a dipole the separation distance is held fixed.

Suppose the dipole is placed in a uniform electric field (i.e., $\vec{E}$ is the same everywhere in space).

The electric dipole



The electric dipole



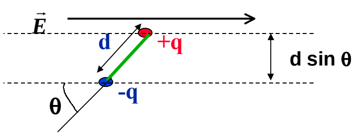
What is the total force acting on the dipole?

Zero, because the force on the two charges cancel: both have magnitude qE. The center of mass does not accelerate.

But the charges start to move. How? Why?

There's a torque because the forces aren't colinear

The electric dipole

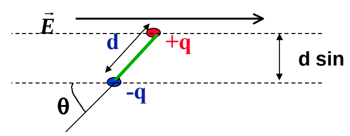


The torque is τ=(magnitude of force)(moment arm)

= (qE)(d sin θ)

It's nice to write this in vector form.

The electric dipole



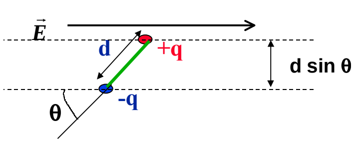
The torque is τ=(magnitude of force)(moment arm)

= (qE)(d sin θ) = p sin θ E,
where p = qd

It's nice to write this in vector form.

Then the torque is $\vec{\tau} = \vec{p} \times \vec{E}$

The electric dipole



We define the dipole moment to be the vector $\vec{p}$ whose magnitude is qd, and which points from the negative charge to the positive.

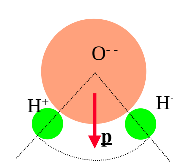
Then the torque is $\vec{\tau} = \vec{p} \times \vec{E}$

which has an associated potential energy $U = -\vec{p} \cdot \vec{E}$

Example: Water Molecule

Torque due to a field:

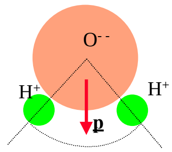
The dipole moment of a water molecule is 6.23×10^{-30} C.m. One mole of water is placed in a uniform E field of 3×10^{5} N/C. What is the magnitude of the torque on a molecule when it is oriented 60^{0} from the field?



Example: Water Molecule

Torque due to a field:

The dipole moment of a water molecule is 6.23×10^{-30} C.m. One mole of water is placed in a uniform E field of 3×10^5 N/C. What is the magnitude of the torque on a molecule when it is oriented 60^0 from the field?



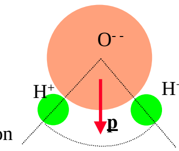
$\boldsymbol{\tau} = \mathbf{p} \times \mathbf{E}$

$\tau = pE \sin(60^o) = 1.62 \times 10^{-24}$ N.m

Example: Water Molecule

Work done by a field:

How much work must be done on this quantity of water to rotate all the dipoles from $\theta = 0^0$ to $\theta = 135^0$?



For a dipole, $U = -\mathbf{p} \cdot \mathbf{E}$ Thus under rotation

$\Delta U_{molecule} = U(\theta = 135^0) - U(\theta = 0^0)$

$= [-pE \cos(135^0)] - [-pE \cos(0^0)] = 3.19 \times 10^{-24}$ J

To increase the potential energy you must do positive work.

Total work is $W = (6.02 \times 10^{23}$ molecules/mole$)(3.19 \times 10^{-24}$ J$)$
$= 1.92$ J