Electric Potential

Chapter 25

Electric Potential Energy

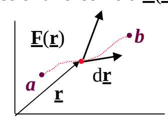
The Electric Potential

Equipotential Surfaces

Calculating the Electric Field from the Potential

Electric Potential Energy

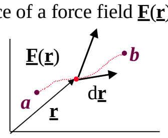
If a particle • is moved along a path from a point ***a*** to a point ***b*** in the presence of a force field **F**(**r**), then the force does **work** on the particle:



Work $= W = \int_a^b \vec{F} \bullet d\vec{r}$

Electric Potential Energy

If a particle • is moved along a path from a point ***a*** to a point ***b*** in the presence of a force field **F**(**r**), then the force does **work** on the particle:



Work $= W = \int_a^b \vec{F} \bullet d\vec{r}$

Some forces are **conservative**: for these, the work depends only on the positions of the end points but not on the particular path connecting them. For conservative forces, one can define a **potential energy U(r)**:

$$U(b) - U(a) = -W = -\int_a^b \vec{F} \bullet d\vec{r}$$

The Electric Potential Energy

Choosing an arbitrary reference point $\mathbf{r}_0$ at which $U(\mathbf{r}_0)=0$, the potential energy is:

$$U(x,y,z) = -\int_{\vec{r}_0}^{\vec{r}} \vec{F} \bullet d\vec{r}$$

This can be inverted:

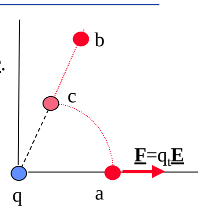
$$\vec{F}(x,y,z) = -(\hat{\mathbf{i}}\, \partial/\partial x + \hat{\mathbf{j}}\, \partial/\partial y + \hat{\mathbf{k}}\, \partial/\partial z) U(x,y,z)$$

The electric force is $1/r^2$, just like that of gravity, hence it is conservative. We can find the potential energy in just the same way.

The Electric Potential Energy

Place a point charge q at the origin. Find the potential energy of a test charge q_t at position **r**.

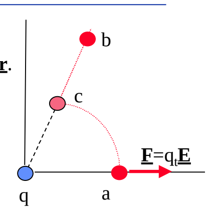
First find the work done by q's field when q_t is moved from a to b on the path a-c-b.



The Electric Potential Energy

Place a point charge q at the origin. Find the potential energy of a test charge q_t at position **r**.

First find the work done by q's field when q_t is moved from a to b on the path a-c-b.



W = W(a to c) + W(c to b)

W(a to c) = 0 because on this path $\vec{F} \perp d\vec{r}$

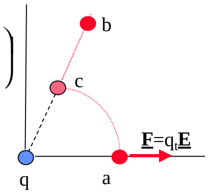
W(c to b)= $\int_{\vec{r}_c}^{\vec{r}_b} q_t \vec{E}(\vec{r}) \bullet d\vec{r} = \int_{r_c}^{r_b} q_t E(r) dr = kq_t q \int_{r_c}^{r_b} \frac{dr}{r^2}$

hence **W** $= -kq_t q\left(\frac{1}{r_b} - \frac{1}{r_c}\right) = -kq_t q\left(\frac{1}{r_b} - \frac{1}{r_a}\right)$

The Electric Potential Energy

Use this to define

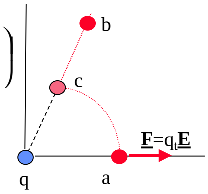
$$U(\vec{r}_b) - U(\vec{r}_a) = -W = kq_t q\left(\frac{1}{r_b} - \frac{1}{r_a}\right)$$



The Electric Potential Energy

Use this to define

$$U(\vec{r}_b) - U(\vec{r}_a) = -W = kq_t q\left(\frac{1}{r_b} - \frac{1}{r_a}\right)$$



From this it's natural to choose the zero of potential energy to be when $r \to \infty$. So letting a be the point at infinity, and dropping the subscript b, we get the potential energy:

$$U(\vec{r}) = k\frac{q_t q}{r}$$

when the separation between the two charges is r.

EXAMPLE: What is the potential energy between two protons in the Uranium nucleus ?

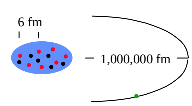
The 92 protons in the nucleus of ^{238}U are on average about 6 fm apart.

$q_1 = q_2 = 1.6 \times 10^{-19}$ C

$$U = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r}$$

$$= \frac{(8.99\times10^9\, Nm^2/C^2)(1.6\times10^{-19}\,C)^2}{6.0\times10^{-15}\,m}$$

$$= 3.8\times10^{-14}\,J = 2.4\times10^5\,eV = 240\,keV$$

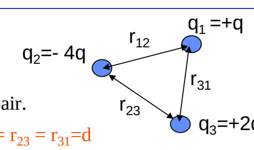


This is a huge energy. The atomic binding energy of an electron is only about 1 eV. Why doesn't the Coulomb force split the nucleus apart?

Because the two protons are held together in the nucleus by the attractive "strong nuclear force."

The Electric Potential Energy

This is a scalar. With more than two charges the total potential energy is simply a sum over the energy of each pair.



Here are 3 charges. Take $r_{12} = r_{23} = r_{31} = d$

$$U = \frac{1}{4\pi\varepsilon_0}\left[\frac{(+q)(-4q)}{d} + \frac{(+q)(+2q)}{d} + \frac{(-4q)(+2q)}{d}\right]$$

$$= -\frac{10q^2}{4\pi\varepsilon_0 d}$$

The Electric Potential

The electric potential V is defined as the *electric potential energy per unit charge*. That is, just divide out the test charge.

Definition is completely analogous to the Electric field, $E(r) = F(r)/q_{test}$

$$V(\vec{r}) = \frac{U(\vec{r})}{q_{test}}$$

Units:
1 Volts = 1 Joules / Coulomb

The Electric Potential

The electric potential V is defined as the *electric potential energy per unit charge*. That is, just divide out the test charge.

Definition is completely analogous to the Electric field, $E(r) = F(r)/q_{test}$

$$V(\vec{r}) = \frac{U(\vec{r})}{q_{test}}$$

Units:
1 Volts = 1 Joules / Coulomb

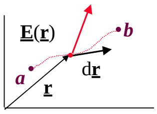
For a point charge at the origin the potential is $V(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\frac{q}{r}$

V is related to the electric potential energy U .... but don't confuse them!

A convenient unit of energy is given by the potential energy U which a charge e has if its potential is 1V:
$U = (1.6 \times 10^{-19}\,C)(1\,V) = 1.6 \times 10^{-19}\,J = 1$ Electron-volt $= 1$ eV

The Electric Potential

There is an easy relationship between the the electric potential energy and the electric potential: just divide out q_{test}. This just replaces each occurence of force $\underline{\mathbf{F}}(\underline{\mathbf{r}})$ by the field $\underline{\mathbf{E}}(\underline{\mathbf{r}})$.



$$\Delta V = \Delta U / q_{test} = -\frac{1}{q_{test}}\int_a^b \vec{F}\bullet d\vec{r} = -\int_a^b \vec{E}\bullet d\vec{r}$$

Choosing $V(\mathbf{r}_0) = 0$, at some $\mathbf{r}_0$:

$$V(x,y,z) = -\int_{\vec{r}_0}^{\vec{r}} \vec{E}\bullet d\vec{r}$$

and...

$$\vec{E}(x,y,z) = -(\hat{i}\,\partial/\partial x + \hat{j}\,\partial/\partial y + \hat{k}\,\partial/\partial z)V(x,y,z)$$

Example: An α particle (q =+2e) in a nuclear accelerator moves from a terminal at +6.5 x 10^5 V to another at 0 V. What is the change in kinetic energy of the particle ?

Example: An α particle (q =+2e) in a nuclear accelerator moves from a terminal at +6.5 x 10^5 V to another at 0 V. What is the change in kinetic energy of the particle ?

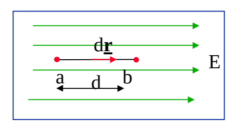
This is like the simplest case of dropping a ball. Associated with the change in potential is some change in potential energy. That potential energy loss is all converted into kinetic energy.

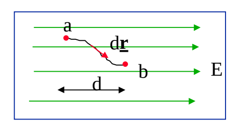
$\Delta U = U_b - U_a = q(V_b - V_a)$
$= (2 \times 1.6\times10^{-19}\,C)(0 - 6.5 \times 10^5\,V)$
$= -2.1 \times 10^{-13}$ J

so the change in kinetic energy is $+2.1 \times 10^{-13}$ J.

Example: Electric potential of a uniform electric field

$$\Delta V = V_b - V_a = -\int_a^b \vec{E}\bullet d\vec{r}$$
$$= -\int_a^b E\,dx = -Ed$$

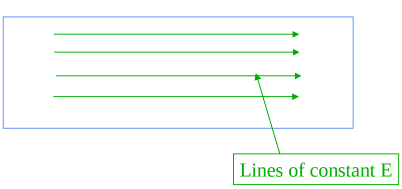




A positive charge would be pushed from regions of high potential to regions of low potential.

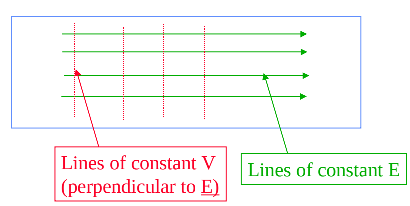
Equipotential surfaces

We can make graphical representations of the electric potential in the same way as we have created for the electric field:



Equipotential surfaces

We can make graphical representations of the electric potential in the same way as we have created for the electric field:

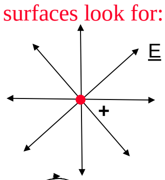


Equipotential plots are like contour maps of hills and valleys.

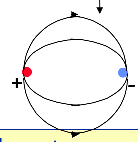
Equipotential surfaces

How do the equipotential surfaces look for:

(a) A point charge?



(b) An electric dipole?



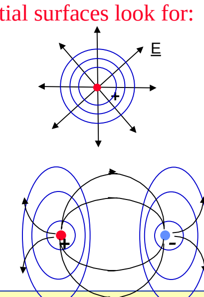
Equipotential plots are like contour maps of hills and valleys.

Equipotential surfaces

How do the equipotential surfaces look for:

(a) A point charge?

(b) An electric dipole?



Equipotential plots are like contour maps of hills and valleys.

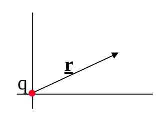
Potential due to several charges

- Just add up the contribution from each charge.
- This <u>Principle of Superposition</u> applies to both the electric field and the electric potential. But it's much easier to apply for a scalar quantity such as the potential.
- This gives a very easy way to calculate the electric field of complicated charge distributions: first find the total potential V(**r**), and then calculate **E** from V.

Example: point charge

Put a point charge q at the origin.

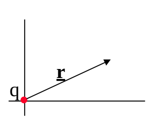
Find V(**r**): here this is easy: $V(\vec{\mathbf{r}}) = k\frac{q}{r}$



Example: point charge

Put a point charge q at the origin.

Find V(**r**): here this is easy: $V(\vec{\mathbf{r}}) = k\frac{q}{r}$



Then find **E**(**r**) from the derivatives:

$$\vec{\mathrm{E}}(\vec{\mathbf{r}}) = -(\hat{\imath}\,\partial/\partial x + \hat{\jmath}\,\partial/\partial y + \hat{k}\,\partial/\partial z)V(x, y, z)$$

Derivative: $\frac{\partial}{\partial x}\frac{1}{r} = \frac{\partial}{\partial x}\frac{1}{\sqrt{x^2+y^2+z^2}} = -\frac{1}{2}\frac{2x}{(x^2+y^2+z^2)^{3/2}}$

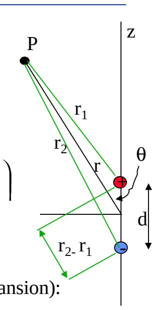
So: $\vec{\mathbf{E}}(\vec{\mathbf{r}}) = kq\frac{x\hat{\mathbf{i}}+y\hat{\mathbf{j}}+z\hat{\mathbf{k}}}{r^3} = kq\frac{\vec{\mathbf{r}}}{r^3} = kq\frac{\hat{\mathbf{r}}}{r^2}$

Example: a dipole

$$V(\vec{\mathbf{r}}) = V_1 + V_2 = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r_1} + \frac{-q}{r_2}\right)$$

$$= \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{|\vec{r} - d\hat{\mathbf{k}}/2|} - \frac{1}{|\vec{r} + d\hat{\mathbf{k}}/2|}\right)$$

$$= \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{\sqrt{x^2+y^2+(z-d/2)^2}} - \frac{1}{\sqrt{x^2+y^2+(z+d/2)^2}}\right)$$



From this it's easy to take the derivatives to calculate **E**.

In the limit of d<<r this simplifies (see binomial expansion):

$V(\vec{\mathbf{r}}) \cong \frac{p}{4\pi\varepsilon_0}\frac{z}{r^3}$ where p=qd, the dipole moment!

Example: a dipole

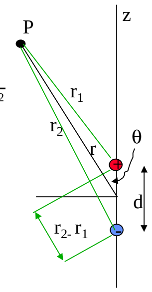
It's really no harder to use the exact expression.
But here find **E** from the approximate V.

$$\frac{\partial}{\partial x}\frac{z}{r^3} = \frac{\partial}{\partial x}\frac{z}{(x^2+y^2+z^2)^{3/2}} = -\frac{3}{2}\frac{2xz}{(x^2+y^2+z^2)^{5/2}}$$

$$\frac{\partial}{\partial z}\frac{z}{r^3} = \frac{1}{r^3} - \frac{3}{2}\frac{2zz}{(x^2+y^2+z^2)^{5/2}}$$

With a similar expression for the y derivative,

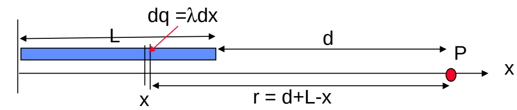
$$\vec{\mathbf{E}}(\vec{\mathbf{r}}) = kp\frac{(3xz\hat{\mathbf{i}} + 3yz\hat{\mathbf{j}}) + (3z^2 - r^2)\hat{\mathbf{k}}}{r^5}$$



This was much easier than adding vector components.

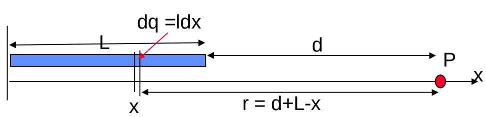
Example: a line of charge

A charge density per unit length λ=400 mC/m stretches for 10 cm.
Find the electric potential at a point 15 cm from one end.



Break the charge into little bits: say a length dx at position x.
The contribution due to this bit at P is:

$$dV = \frac{k(\lambda dx)}{r} = \frac{k\lambda}{d+L-x}dx$$



$$V = \int_0^L \frac{\lambda}{4\pi\varepsilon_0}\frac{1}{[(d+L)-x]}dx$$

$$= \{-\ln[(d+L)-x]\}_0^L = \frac{\lambda}{4\pi\varepsilon_0}\ln(\frac{d+L}{d})$$

$$= \frac{(0.44\times10^{-6}C/m)}{4\pi[8.9\times10^{-12}C^2/N-m^2]}\ln(\frac{0.25m}{0.15m})$$

$$= 1.84\times10^3 V$$

Example: a disk of charge

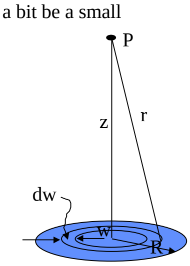
Suppose the disk has radius R and a charge per unit area σ.
Find the potential and electric field at a point up the z axis.
Divide the object into small elements of charge and find the potential dV at P due to each bit. So here let a bit be a small ring of charge width dw and radius w.

$$dq = \sigma 2\pi w dw$$

$$dV = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r} = \frac{1}{4\pi\varepsilon_0}\frac{\sigma 2\pi w dw}{\sqrt{w^2+z^2}}$$

$$V = \int dV = \frac{\sigma}{2\varepsilon_0}\int_0^R (w^2+z^2)^{-1/2} w dw$$

$$V = \frac{\sigma}{2\varepsilon_0}(\sqrt{R^2+z^2} - z)$$

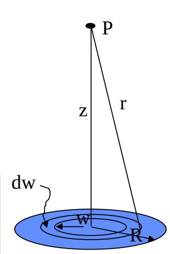


Example: a disk of charge

$$V(z) = \frac{\sigma}{2\varepsilon_0}(\sqrt{R^2+z^2} - z)$$

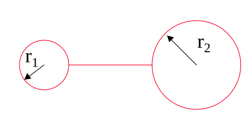
By symmetry one sees that E_x=E_y=0 at P.
Find E_z from

$$E_z = -\frac{\partial V}{\partial z} = -\frac{\sigma}{2\varepsilon_0}\left(\frac{z}{\sqrt{R^2+z^2}} - 1\right)$$

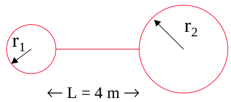


This is easier than integrating over the components of vectors. Here we integrate over a scalar and then take partial derivatives.

Example: two charged spherical conductors are connected by a long conducting wire of length L = 4 m. A total charge of q is placed on this combination of two spheres, where q = 1.0 x 10^{-7}C. Sphere 1 has a radius of r_1 = 10 cm and 2 has a radius of r_2 = 15 cm. Find the tension in the wire.



Example: two charged spherical conductors are connected by a long conducting wire of length L = 4 m. A total charge of q is placed on this combination of two spheres, where q = 1.0 x 10^{-5}C. Sphere 1 has a radius of r_1 = 10 cm and 2 has a radius of r_2 = 15 cm. Find the tension in the wire.



$V_1 = V_2$!

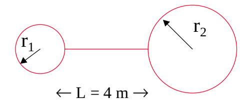
The tension in the wire is force of repulsion between the two spheres i.e. we'll have to use Coulomb's Law after the spheres are converted to equivalent point charges, q_1 & q_2.

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_o}\frac{q}{r} = \frac{1}{4\pi\varepsilon_o}\frac{4\pi r^2\sigma}{r}$$

Voltage at the surface of a charged sphere of radius R is the same as the voltage of the same charge as a point a distance R away.

$$V(\vec{r}) = \frac{r\sigma}{\varepsilon_o}$$

Example: two charged spherical conductors are connected by a long conducting wire of length L = 4 m. A total charge of q is placed on this combination of two spheres, where q = 1.0 x 10^{-5}C. Sphere 1 has a radius of r_1 = 10 cm and 2 has a radius of r_2 = 15 cm. Find the tension in the wire.



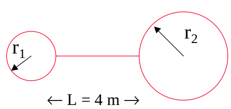
$$V_1 = \frac{r_1\sigma_1}{\varepsilon_o}$$

$$V_2 = \frac{r_2\sigma_2}{\varepsilon_o}$$

$$q = q_1 + q_2 = 4\pi r_1^2\sigma_1 + 4\pi r_2^2\sigma_2$$

$$r_2\sigma_2 = \frac{q}{4\pi r_2} - \sigma_1\frac{r_1^2}{r_2}$$

Example: two charged spherical conductors are connected by a long conducting wire of length L = 4 m. A total charge of q is placed on this combination of two spheres, where q = 1.0 x 10^{-5}C. Sphere 1 has a radius of r_1 = 10 cm and 2 has a radius of r_2 = 15 cm. Find the tension in the wire.



$$V_1 = V_2$$

$$r_1\sigma_1 = r_2\sigma_2$$

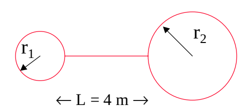
$$r_2\sigma_2 = \frac{q}{4\pi r_2} - \sigma_1\frac{r_1^2}{r_2}$$

$$\sigma_1 = \frac{q}{4\pi r_1(r_2 + r_1)}$$

$$q_1 = 4\pi r_1^2\sigma_1 = q\frac{r_1}{(r_2 + r_1)}$$

$$q_2 = q\frac{r_2}{(r_2 + r_1)}$$

Example: two charged spherical conductors are connected by a long conducting wire of length L = 4 m. A total charge of q is placed on this combination of two spheres, where q = 1.0 x 10^{-5}C. Sphere 1 has a radius of r_1 = 10 cm and 2 has a radius of r_2 = 15 cm. Find the tension in the wire.



$$F = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}$$

$$F = \frac{1}{4\pi\varepsilon_0}\frac{q\frac{r_1}{(r_2+r_1)}\,q\frac{r_2}{(r_2+r_1)}}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{q^2 r_1 r_2}{(r_2 + r_1 + L)^2(r_2 + r_1)^2}$$

$$F = 0.012\ \text{N}$$