Gauss's Law

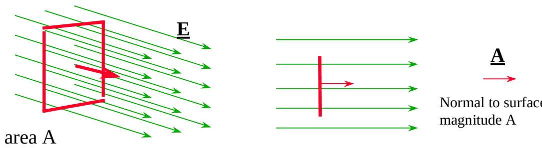
Chapter 24

Gauss's law and electric flux

Gauss's law is based on the concept of flux:

You can think of the flux through some surface as a measure of the number of field lines which pass through that surface.

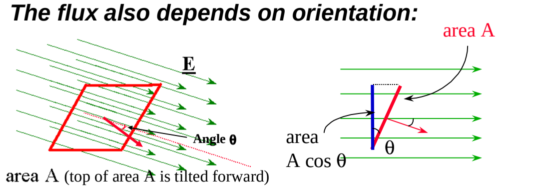
Flux depends on the strength of **E**, on the surface area, and on the relative orientation of the field and surface.



Here the flux is
Φ = E A

Electric flux

The flux also depends on orientation:



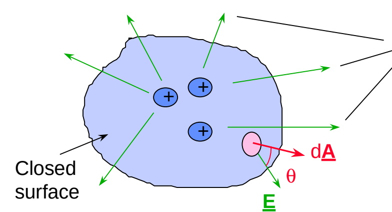
The number of field lines through the tilted surface / equals the number through its projection |. Hence the flux through the tilted surface is simply given by the flux through its projection: E(Acosθ).

Here flux Φ = E A cos θ = E • A

Electric flux

But what if the electric field is not constant? What if it varies (possibly in both magnitude and direction) as a function of **r** ?

This possibility is sketched here for the case of a closed surface.



How do you calculate the flux? Break the surface into bits d**A** .

Electric flux

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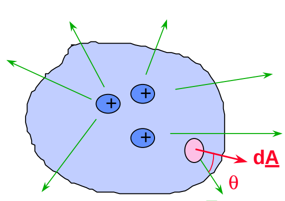

dA is a small area element, like ΔA.
Flux through dA is
$d\Phi = E\, dA \cos\theta$
$d\Phi = \mathbf{E} \cdot d\mathbf{A}$

So how do we find the TOTAL Flux?

Electric flux

But what if the electric field is not constant? What if it varies (possibly in both magnitude and direction) as a function of **r** ?

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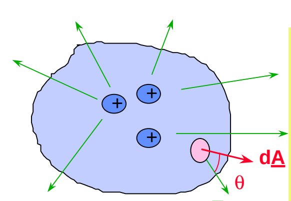

We have to sum all the $d\Phi$'s over the entire surface: $\Phi = \Sigma\, d\Phi$
For accuracy the *dA*'s and thus $d\Phi$'s must be very small.

Electric flux

But what if the electric field is not constant? What if it varies (possibly in both magnitude and direction) as a function of **r** ?

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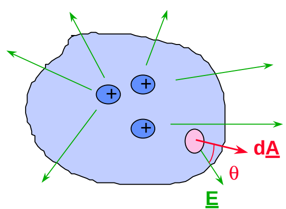


$$\Phi = \lim_{dA \to 0} \sum_{closed\ surface} d\Phi$$

$$= \lim_{dA \to 0} \sum_{closed\ surface} \vec{E} \cdot d\vec{A}$$

$$= \oint_{closed\ surface} \vec{E} \cdot d\vec{A}$$

The loop means the integral is over a closed surface.

Gauss's Law

Electric flux through any closed surface = (charge inside) / ε_0

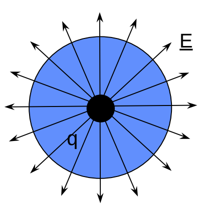
Hence, Gauss' Law states:

$$\Phi = \oint d\Phi = \oint \underline{E} \cdot \underline{dA} = \frac{\sum_{inside} q}{\varepsilon_0}$$

This is always true. It's sometimes useless, but often a very easy way to find the electric field (for highly symmetric cases).

Apply Gauss' s law to a point charge

Consider a positive point charge q. Define a Gaussian surface (i.e. a closed surface) which is a sphere of radius r. By symmetry, the lines of **E** must be radially outwards, with magnitude depending only on r.



$$\frac{q}{\varepsilon_0} = \oiint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \oiint EdA = E \oiint dA$$

using $\oiint dA = \text{total area} = 4\pi r^2$

gives $E = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2}$

→ ***Coulomb's Law!***

Symmetry

- Apply Gauss's Law to a point charge and what do you get? Answer: Coulomb's Law!
- We used the fact that a point charge in space is spherically symmetric.
- Gauss's Law is always true, but is only useful for problems with usable symmetry.

Is Gauss's Law more fundamental than Coulomb's Law?

- Maybe? Here we derived Coulomb's law for a point charge from Gauss's law.
- One can instead derive Gauss's law for a general (even very nasty) charge distribution from Coulomb's law. The two laws are equivalent.
- Gauss's law gives us an easy way to solve very symmetric problems in electrostatics.
- Gauss's law also gives us great insight into the electric fields in and on conductors and within voids inside metals.
- Gauss's law has applications in electricity, magnetism, and even gravity. Mathematically, it applies fundamentally to vector fields and their potentials. Mathematically, Gauss's law is very fundamental.

Gauss's Law

The total flux within a closed surface **is proportional to the enclosed charge.**

$$\Phi = \oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{Q_{enclosed}}{\varepsilon_0}$$

Gauss's Law is always true, but is only useful for problems with usable symmetry.

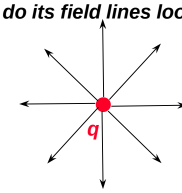
Symmetry and the Electric Field

Can we figure out how the field varies with distance from the field lines and the symmetry?

Symmetry and the Electric Field

Can we figure out how the field varies with distance from the field lines and the symmetry?

Look at a point charge:
- How do its field lines look?



This is a 2D picture. Next we'll try to get a picture of a 3D piece.

Field lines point out (or in) radially in all directions (3D)

Symmetry and the Electric Field

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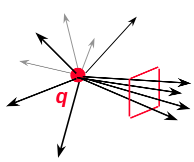
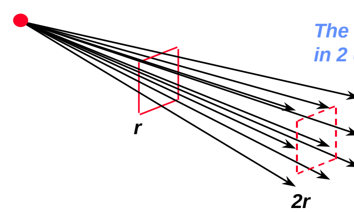


Recall that the magnitude of E is related to the density of field lines per unit area.

Field lines point out (or in) radially in all directions (3D)

Symmetry and the Electric Field



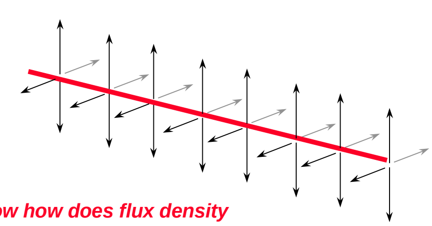
The lines spread in 2 directions.

How does the number of field lines per unit area vary with distance?

- Inverse - square law: $E \propto \frac{1}{r^2}$

Symmetry and the Electric Field

Line of charge:

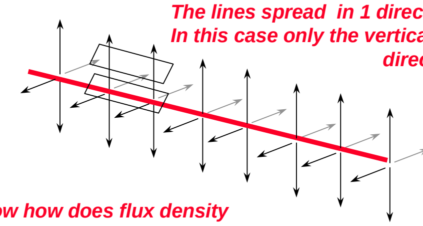


Now how does flux density vary with distance?

Symmetry and the Electric Field

Line of charge:

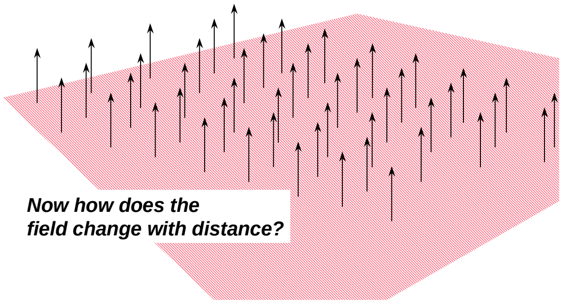
The lines spread in 1 direction. In this case only the vertical direction.



Now how does flux density vary with distance?

Symmetry and the Electric Field

Sheet of charge:

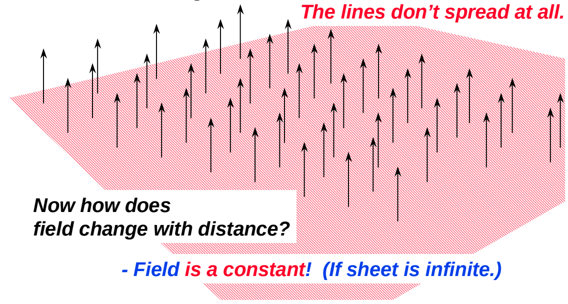


Now how does the field change with distance?

Symmetry and the Electric Field

Sheet of charge:

The lines don't spread at all.



Now how does field change with distance?

- Field is a constant! (If sheet is infinite.)

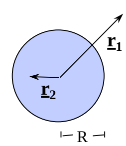
Applications of Gauss's Law

Gauss's Law does what we just did above, but does it rigorously.

We are now going to look at various charged objects and use Gauss's law to find the field distribution.

Problem: Sphere of Charge Q

A charge Q is uniformly distributed through a sphere of radius R. What is the electric field as a function of **r**? Find **E** at $\mathbf{r}_1$ and $\mathbf{r}_2$.



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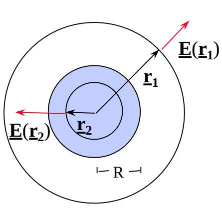
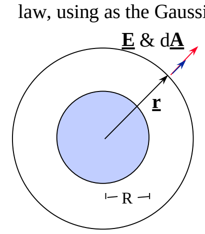


Use symmetry!

This is spherically symmetric. That means that **E**(**r**) is radially outward, and that all points at a given radius (|**r**|=r) have the same magnitude of field.

Problem: Sphere of Charge Q

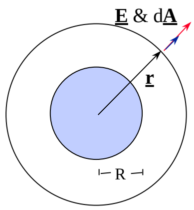
First find **E**(**r**) at a point outside the charged sphere. Apply Gauss's law, using as the Gaussian surface the sphere of radius r pictured.



What is the enclosed charge? **Q**

Problem: Sphere of Charge Q

First find $\mathbf{E}(\mathbf{r})$ at a point outside the charged sphere. Apply Gauss's law, using as the Gaussian surface the sphere of radius r pictured.



What is the enclosed charge? **Q**

What is the flux through this surface?

$$\Phi = \oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \oint E\, dA$$
$$= E\oint dA = EA = E(4\pi r^2)$$

Gauss: $\Phi = Q_{enclosed}/\varepsilon_o = Q/\varepsilon_o$

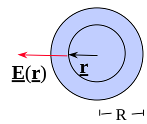
$$Q/\varepsilon_0 = \Phi = E(4\pi r^2)$$

So $\vec{\mathbf{E}}(\vec{\mathbf{r}}) = \frac{1}{4\pi\varepsilon_o}\frac{Q}{r^2}\hat{\mathbf{r}}$

Exactly as though all the charge were at the origin! (for r>R)

Problem: Sphere of Charge Q

Next find $\mathbf{E}(\mathbf{r})$ at a point **inside** the sphere. Apply Gauss's law, using a little sphere of radius r as a Gaussian



What is the enclosed charge?
That takes a little effort. The little sphere has some fraction of the total charge. What fraction?

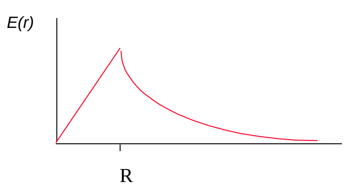
That's given by volume ratio: $Q_{enc} = \frac{r^3}{R^3}Q$

Again the flux is: $\Phi = EA = E(4\pi r^2)$

setting $\Phi = Q_{enc}/\varepsilon_o$ gives $E = \frac{(r^3/R^3)Q}{4\pi\varepsilon_o r^2}$

for $r < R$, $\vec{\mathbf{E}}(\vec{\mathbf{r}}) = \frac{Q}{4\pi\varepsilon_o R^3} r\hat{\mathbf{r}}$

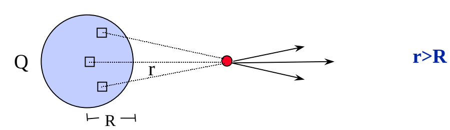
Problem: Sphere of Charge Q



E(r) is proportional to r for r<R
E(r) is proportional to $1/r^2$ for r>R
and E(r) is continuous at R

Problem: Sphere of Charge Q

Look closer at these results. The electric field at ● comes from a sum over the contributions of all the little bits.

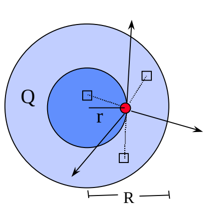


It's obvious that the net $\mathbf{E}$ at this point will be horizontal.
But the magnitude from each bit is different; and it's completely not obvious that the magnitude E just depends on the distance from the sphere's center to the observation point.

Doing this as a volume integral would be HARD.
Gauss's law is EASY.

Problem: Sphere of Charge Q

Now look at an observation point ● inside the sphere. **r<R**



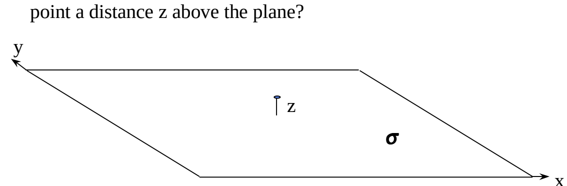
Because of the spherical symmetry, the contributions from the bits outside the radius of ● exactly **cancel** one another!

The field at r is exactly what you would have if all the charge within the radius r were concentrated to a point at the origin.

Doing this as a volume integral would be HARD.
Gauss's law is EASY.

Problem: Infinite charged plane

Consider an infinite plane with a constant charge density σ (which is some number of Coulombs per square meter). What is $\mathbf{E}$ at a point a distance z above the plane?



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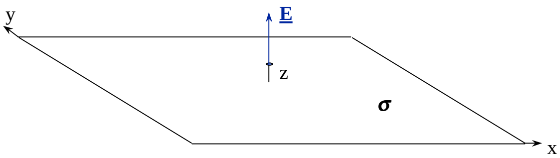
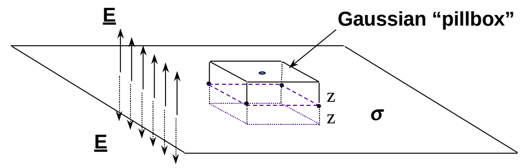


Use symmetry!

The electric field must point straight away from the plane (if σ>0). Maybe the magnitude E depends on z, but the direction is fixed. And **E** is independent of x and y.

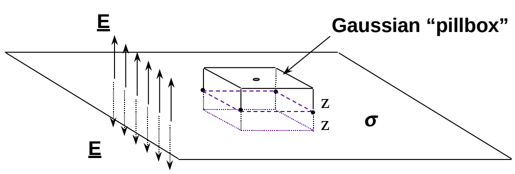
Problem: Infinite charged plane

So choose a Gaussian surface which is a "pillbox" which has its top above the plane and its bottom below the plane, each a distance z from the plane. That way the observation point lies in the top.



Problem: Infinite charged plane

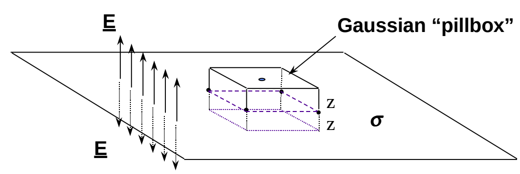
Let the area of the top and bottom be A.



Total charge enclosed by box = Aσ

Problem: Infinite charged plane

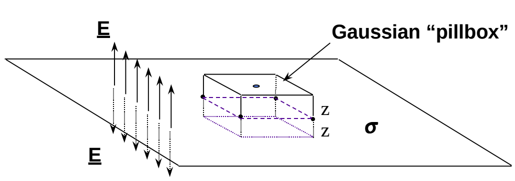
Let the area of the top and bottom be A.



Outward flux through the top: EA
Outward flux through the bottom: EA
Outward flux through the sides: E x (some area) x $\cos(90^0)=0$
So the total flux is: 2EA

Problem: Infinite charged plane

Let the area of the top and bottom be A.

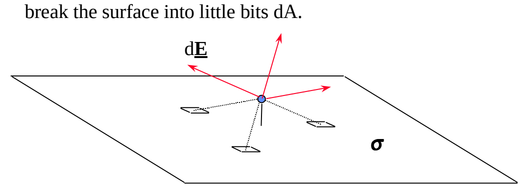


Gauss's law then says that $A\sigma/\varepsilon_0=2EA$ so that **E**$=\sigma/2\varepsilon_0$, outward.
This is constant everywhere in each half-space!

Notice that the area A canceled: this is typical!

Problem: Infinite charged plane

Imagine doing this with an integral over the charge distribution: break the surface into little bits dA.



Doing this as a surface integral would be HARD.
Gauss's law is EASY.

Conductors

- A conductor is a material in which charges can move relatively freely.
- Usually these are metals.
- In a static condition, the charges placed on a conductor will have moved as far from each other as possible - they repel each other.
- In a static situation, the electric field is zero everywhere inside a conductor.

Conductors

Why is <u>E</u>=0 inside a conductor?

Conductors

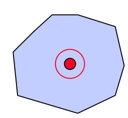
Why is <u>E</u>=0 inside a conductor?

Because conductors are full of free electrons, roughly one per cubic Angstrom. These are free to move. If **<u>E</u>** is nonzero in some region, then the electrons there feel a force -e**<u>E</u>** and start to move.

In an electrostatics problem, the electrons adjust their positions until the force on every electron is zero (or else it would move!). That means when equilibrium is reached, **<u>E</u>**=0 everywhere inside a conductor.

Conductors

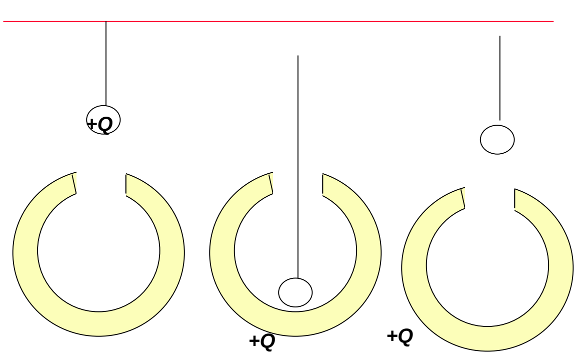
Because E=0 inside, the inside is neutral.



Suppose there is an extra charge ● inside. Gauss's law for the little spherical surface says there would be a nonzero E nearby. But there can't be, within a metal!

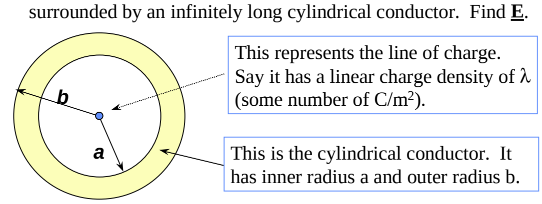
Consequently the interior of a metal is neutral. Any excess charge ends up on the surface.

Electric field in conductors



Problem: Charged coaxial cable

This picture is a cross section of an infinitely long line of charge surrounded by an infinitely long cylindrical conductor. Find **<u>E</u>**.

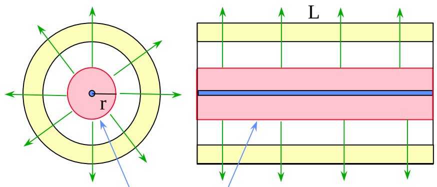


Use symmetry!

Clearly **<u>E</u>** points straight out, and its amplitude depends only on r.

Problem: Charged coaxial cable

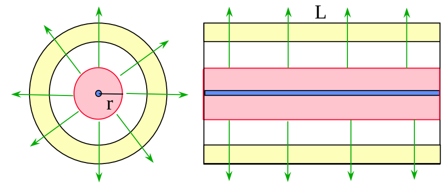
First find E at positions in the space inside the cylinder (r<a).



Choose as a Gaussian surface a cylinder of radius r and length L.

Problem: Charged coaxial cable

First find E at positions in the space inside the cylinder (r<a).



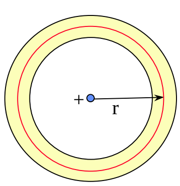
What is the charge enclosed? λL
What is the flux through the end caps? zero ($\cos 90^0$)
What is the flux through the curved face? E x (area) = $E(2\pi rL)$
Total flux = $E(2\pi rL)$
Gauss's law: $E(2\pi rL) = \lambda L/\varepsilon_0$ so $E(r) = \lambda / 2\pi r\varepsilon_0$

Problem: Charged coaxial cable

Now find E at positions within the cylinder (a<r<b).

There's no work to do: within a conductor **E**=0.

Still, we can learn something from Gauss's law.

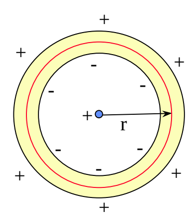


Make the same kind of cylindrical Gaussian surface. Now the curved side is entirely within the conductor, where **E**=0; hence the flux is zero.

Thus the total charge enclosed by this surface must be zero.

Problem: Charged coaxial cable

There must be a net charge per length $-\lambda$ attracted to the inner surface of the metal so that the total charge enclosed by this Gaussian surface is zero.



And since the cylinder is neutral, these negative charges must have come from the outer surface. So the outer surface has a charge density per length of $+\lambda$ spread around the outer perimeter.

So what is the field for r>b? Easy!

Example Problem: Gauss' Law for Gravity

"Gauss' law for gravitation" is $\Phi_g = \oint_{\text{Gaussian Surface}} \vec{g} \cdot d\vec{A} = -4\pi G m_{enclosed}$

In which Φ_g is the net flux of the gravitational field **g** through a gaussian surface that encloses a mass ($m_{enclosed}$). The field **g** is defined to be the acceleration of a test particle on which $m_{enclosed}$ exerts a gravitational force.

Calculate Newton's Law of Gravitation from this.

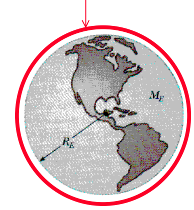
Example Problem: Gauss' Law for Gravity

$$\Phi_g = \oint_{\text{Gaussian Surface}} \vec{g} \cdot d\vec{A} = -4\pi G m_{enclosed} \qquad F = m_1 a = G\frac{m_1 m_2}{r^2}$$

$$\Phi_g = \oint_{\text{Gaussian Surface}} \vec{g} \cdot d\vec{A} = -\oint_{\text{Gaussian Surface}} g\, dA$$

$$\Phi_g = -\oint_{\text{Gaussian Surface}} g\, dA = -g \oint_{\text{Gaussian Surface}} dA$$

$$= -gA = -g(4\pi r^2)$$



$\Phi_g = -4\pi G m_{enclosed} = -4\pi G m_2$

$\Phi_g = -4\pi G m_2 = -g(4\pi r^2) \rightarrow$ accel. $= g = \frac{Gm_2}{r^2} = a$